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Numerical solution of the Helmholtz equation in Optics

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It is a difficult task to solve the Helmholtz equation $(\Delta + K^2)\psi(r) = 0$ for large values of K, since it is practically impossible to use either the finite-difference method or the Green's function method. A new method is given which leads to an attractive finite-difference scheme, explicit but nevertheless stable provided that a rather unrestrictive condition is satisfied. The new method is tested in the case of laser beam propagation, and it gives excellent results.

1. INTRODUCTION

Let us consider an unpolarized light beam propagating along the Oz axis of $\mathbb{R}^3$ and characterized by a complex scalar $\psi(r)$ $(r = (x, y, z) = (\rho, z))$, which is the solution of the Helmholtz equation

$$(\Delta + K^2)\,\psi(r) = 0, \qquad K^2 = K_0^2(1 + \epsilon\mu(r))^2, \tag{1}$$

where Δ is the Laplacian, K_0 is the wavenumber, $n(r) = 1 + \epsilon\mu(r)$ denotes the index of refraction, and the quantity ϵ measures the deviation of the index from unity.

The following conditions guarantee the existence of a solution of Eq. (1) and its uniqueness.

$$(\psi(r))_{z=0} = \psi^0(\rho); \tag{1a}$$

$$\psi(\rho, z) = O(|\,\rho\,|^{-\alpha}), \text{ as } \rho \to \infty, \qquad \alpha \geqslant 2 \quad \forall z \geqslant 0. \tag{1b}$$

In optics, K_0 is about 10^5 cm^{-1}, which leads to numerical difficulties that are readily apparent for the simpler case $\epsilon = 0$:

$$(\Delta + K_0^2)\,\psi(r) = 0. \tag{2}$$

Indeed, there exist two main possibilities for solving (2):

(a) *The Green's function method.* The Green's function for the half-space $z > 0$, null on the $z = 0$ plane, is [1]

$$G(r, r') = \frac{e^{iK_0|r-r'|}}{|\,r - r'\,|} - \frac{e^{iK_0|\hat{r}-r'|}}{|\,\hat{r} - r'\,|}; \qquad r = (x, y, z), \quad \hat{r} = (x, y, -z), \quad i = \sqrt{-1},$$

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and it is easy to show that the solution of Eq. (1) for the boundary condition (1a) is [1]

$$\psi(R) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_{z=0} \psi^0(\rho')\frac{e^{iK_0R}}{R}\,dx'\,dy'; \qquad R^2 = (x-x')^2 + (y-y')^2 + z^2.$$

For $K_0 \sim 10^5$, the integrand on the right-hand side of the last equation is a highly oscillatory function, and there does not exist a quadrature formula that gives a correct result.

(b) *The finite-difference method.* Equation (2) can be written in the form

$$\frac{\partial^2\psi(r)}{\partial z^2} = L\psi(r); \qquad L = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + K_0^2\right) \stackrel{\mathrm{d}}{=} -(\Delta_\perp + K_0^2), \qquad (2')$$

where $\Delta_\perp$ is the transverse Laplacian. One could consider using the Taylor series expansion of $\psi(\rho, z+h) + \psi(\rho, z-h)$, which depends only on $(\partial^{2p}/\partial z^{2p})\,\psi(r)$, that is, from (2′), on $L^p\psi(r)$; but, by a rearrangement of terms with respect to increasing powers of $\Delta_\perp{}^p\psi(r)$, it is easy to show that one obtains a power series in $K_0^{2p}h^{2p}/(2p)!$ numerically unsuitable except for unacceptably small values of h.

Here, we give a numerical method for solving Eqs. (1) and (2) with boundary conditions (1a) and (1b). Most of the work on the Helmholtz equation in optics concerns an asymptotic solution [2], and a list of various attempts can be found in [3].

2. Paraxial Approximation to the Solution of Eq. (2)

Formally,[1] the solutions of Eq. (2′) are

$$\psi(z) \propto \sin(\text{const} + z(K_0^2 + \Delta_\perp)^{1/2})\psi_0(\rho),$$

and these lead to the recurrence relation

$$\psi^{n+1} + \psi^{n-1} = 2\cos(h(K_0^2 + \Delta_\perp)^{1/2})\,\psi^n; \qquad \psi^n = \psi(\rho, nh),$$

which, in first-order approximation in $\Delta_\perp$, valid for paraxial conditions (slow—transverse variations on the wavelength scale, slow variation of K) becomes

$$\psi^{n+1} + \psi^{n-1} = (2\cos K_0h - (h/K_0)\sin K_0h\Delta_\perp)\,\psi^n.$$

Now, with $\Delta_\perp$ replaced by the five-point Poisson operator, this becomes

$$\psi_{jl}^{n+1} + \psi_{jl}^{n-1} = 2\cos K_0h\psi_{jl}^n - b\sin K_0h\{\psi_{j+1,l}^n + \psi_{j-1,l}^n + \psi_{j,l+1}^n + \psi_{j,l-1}^n - 4\psi_{jl}^n\}, \qquad (3)$$

with $b = h/K_0h_\rho{}^2$, where h_ρ is the step size in the directions Ox and Oy.

[1] I am indebted to a referee for this elegant formal introduction of Eq. (3).

In Appendix 1, we prove Eq. (3) on a less formal basis and we also give higher-order approximations, but in the text we discuss only Eq. (3).

2.1. *Properties of Eq.* (3)

In this section, we consider the three points: local truncation error, stability, and initial data.

The order of the local truncation error is easy to evaluate, since it follows from Appendix 1 that, with respect to b, this order is $O(b^2)$, while as is well known, the Poisson operator is an $O(h_\rho^2)$ approximation of $\Delta_\perp$.

To check the stability of (3), we use a method previously described [4]. Assuming that the indices j, l, n take the values $1,..., J$, $1,..., L$, $1,..., N$, respectively, one introduces a new index r taking values $1,..., JLN$ and defined by the relation

$$r = n + N(l-1) + NL(j-1); \qquad r = 1, 2,..., JLN.$$

Then $\psi_r \equiv \psi_{jl}^n$, and Eq. (3) becomes a simple difference equation

$$\begin{aligned}\psi_{r+1} + \psi_{r-1} &= 2\cos K_0 h \psi_r \\ &\quad - b \sin K_0 h(\psi_{r+NL} + \psi_{r-NL} + \psi_{r+N} + \psi_{r-N} - 4\psi_r). \end{aligned} \tag{3'}$$

The z-transformation [5] of (3′) gives the characteristic equation

$$z^{r+1} + z^{r-1} = 2\cos K_0 h z^r - b \sin K_0 h(z^{r+NL} + z^{r-NL} + z^{r+N} + z^{r-N} - 4z^r),$$

and dividing by z^r:

$$P(z) \equiv z^2 - \{2\cos K_0 h - b \sin K_0 h(z^{NL} + z^{-NL} + z^N + z^{-N} - 4)\}z + 1 = 0. \tag{4}$$

The solution of (3′) is stable if and only if all roots z_i of the polynomial $P(z)$ satisfy $|z_i| \leqslant 1$ and if any roots of modulus 1 are simple. In [4], we introduced a Nyquist-like criterion, using the rational function $Q(z) = z^{-2}(P(z) - z^2)$, and we proved that the characteristic equation has no root outside the unit circle if the curve $Q(e^{i\theta})$, $0 \leqslant \theta \leqslant 2\pi$, does not encircle the point -1. In the present case, one has

$$Q(z) = z^{-2}\{1 - [2\cos K_0 h - b \sin K_0 h(z^{NL} + z^{-NL} + z^N + z^{-N} - 4)]z\}.$$

The real and imaginary parts of $Q(e^{i\theta})$ are

$$u(\theta) = \cos 2\theta - 2\cos\theta a(\theta); \qquad v(\theta) = -(\sin 2\theta - 2\sin\theta a(\theta)),$$

with

$$a(\theta) = \cos K_0 h + 2b \sin K_0 h\left(\sin^2\frac{NL\theta}{2} + \sin^2\frac{N\theta}{2}\right).$$

We also proved [4] that if the θ_i are the zeros of $v(\theta)$, the curve $Q(e^{i\theta})$ does not encircle the point -1 if all the quantities $1 + u(\theta_i)$ have the same sign. Here, one has $v(\theta) = 0$

for $\theta = 0, \theta = \pi, \theta = \theta_0$, where the θ_0 are the roots of the equation $\cos\theta_0 - a(\theta_0) = 0$ when such roots exist, and then

$$1 + u(0) = 2(1 - \cos K_0 h); \qquad 1 + u(\pi) = 2(1 + a(\pi/2)); \qquad 1 + u(\theta_0) = 0;$$

but one must have $|a(\theta_0)| \leqslant 1$, and since $|a(\theta_0)| \leqslant \sup_{0\leqslant\lambda\leqslant 2} |\cos K_0 h + 2\lambda \sin K_0 h|$, with

$$\lambda = \sin^2 \frac{NL\theta}{2} + \sin^2 \frac{N\theta}{2},$$

it follows that

$$\sup_{0\leqslant\lambda\leqslant 2} |\cos K_0 h + 2\lambda b \sin K_0 h| \leqslant 1. \tag{5}$$

Then $(1 + u(0))(1 + u(\pi)) > 0$ and algorithm (3) is stable. If $\sin K_0 h \cos K_0 h > 0$, the stability condition becomes

$$|\cos K_0 h + 4b \sin K_0 h| \leqslant 1. \tag{5'}$$

Remark 1. In Appendix 1, we prove that $b < 1$.

Remark 2. To check the stability of (3), one could have used the Von Neumann harmonic analysis, which also supplies the stability conditions (5).

One must still discuss the initial condition for starting Eq. (3). In fact, we consider here a Dirichlet problem, and the boundary condition (1a) supplies the data ψ_{jl}^0 on the $z = 0$ plane, but the difference equation (3) requires ψ_{jl}^0 and ψ_{jl}^1.

In theory, using the Green's function, one could compute the successive derivatives $(\partial^p\psi/\partial z^r)_{z=0}$ and obtain ψ_{jl}^1 with a Taylor series expansion, but as discussed in the Introduction, this method is unworkable. However, one can compute an approximation of ψ_{jl}^1 in the following way.

The solution $\psi(r)$ of Eq. (2) is assumed to have the form $\psi(r) = e^{iK_0 z}\varphi(r)$, so that Eq. (2) becomes $\Delta\varphi(r) + 2iK_0(\partial\varphi(r)/\partial z) = 0$, and we further assume (in agreement with the previous paraxial conditions) that

$$\left| K_0 \frac{\partial\varphi}{\partial z} \right| \gg \left| \frac{\partial^2\varphi}{\partial z^2} \right|.$$

Thus, $\varphi(r)$ is a solution of the Schrodinger-like equation

$$2iK_0 \frac{\partial\varphi(r)}{\partial z} + \Delta_\perp \varphi(r) = 0,$$

which for the first step leads to the equation ($\varphi_{jl}^0 = \psi_{jl}^0$)

$$\begin{aligned} \varphi_{jl}^1 &= \varphi_{jl}^0 + \frac{ib}{2}(\varphi_{j+1,l}^0 + \varphi_{j-1,l}^0 + \varphi_{j,l+1}^0 + \varphi_{j,l-1}^0 - 4\varphi_{jl}^0) \\ &= \psi_{jl}^0 + \frac{ib}{2}(\psi_{j+1,l}^0 + \psi_{j-1,l}^0 + \psi_{j,l+1}^0 + \psi_{j,l-1}^0 - 4\psi_{jl}^0). \end{aligned}$$

So, one can start (3) with ψ_{jl}^0 and $\psi_{jl}^1 = e^{iK_0 h}\varphi_{jl}^1$.

2.2. *Conservation of the Transverse Energy Flux*

We now prove that for the problem defined by Eq. (1) or Eq. (2) with the boundary conditions (1a) and (1b), there exists a constant of motion which is the energy flux across transverse planes.

Let Ω be a closed domain in a transverse plane $z = \text{const}$ and let Γ be its boundary with $\bar{\psi}(r)$ the complex conjugate field; Green's theorem gives

$$\int_\Omega (\bar{\psi}(r)\, \Delta_\perp \psi(r) - \psi(r)\, \Delta_\perp \bar{\psi}(r))\, dx\, dy = \int_\Gamma (\bar{\psi}(r)\, \partial_n \psi(r) - \psi(r)\, \partial_n \bar{\psi}(r))\, dl,$$

where dl is the length element on Γ and ∂_n is the normal derivative. Now, if Γ tends to infinity in both directions, the right-hand side of the previous relation is zero from (1b), so that in the transverse plane $z = \text{const}$, one has

$$\int_{z=\text{const}} (\bar{\psi}(r)\, \Delta_\perp \psi(r) - \psi(r)\, \Delta_\perp \bar{\psi}(r))\, dx\, dy = 0,$$

or using (1),

$$\int_{z=\text{const}} \left(\bar{\psi}(r) \frac{\partial^2 \psi(r)}{\partial z^2} - \psi(r) \frac{\partial^2 \bar{\psi}(r)}{\partial z^2}\right) dx\, dy = 0,$$

and finally,

$$\int_{z=\text{const}} \left(\bar{\psi}(r) \frac{\partial \psi(r)}{\partial z} - \psi(r) \frac{\partial \bar{\psi}(r)}{\partial z}\right) dx\, dy = \text{const}. \tag{6}$$

It is easy to see [6] that

$$\frac{iKc}{8\pi} \left(\bar{\psi}(r) \frac{\partial \psi(r)}{\partial z} - \psi(r) \frac{\partial \bar{\psi}(r)}{\partial z}\right)$$

is the density of the transverse energy flux, so that Eq. (6) expresses the conservation of the transverse energy flux.

From a numerical point of view, Eq. (6) is an important relation, since it enables one to test the stability and the quality of computations which should be stopped as soon as (6) is no longer constant. But the problem is to obtain a good approximation of $\partial\psi(r)/\partial z$. For this, it is sufficient to note that the first derivative

$${}^{(1)}\psi_{jl}^n \equiv \left(\frac{\partial \psi(r)}{\partial z}\right)_{jl}^n$$

also satisfies the difference equation (3):

$$\begin{aligned} {}^{(1)}\psi_{jl}^{n+1} + {}^{(1)}\psi_{jl}^{n-1} &= 2 \cos K_0 h\ {}^{(1)}\psi_{jl}^n - b \sin K_0 h \\ &\quad \times ({}^{(1)}\psi_{j+1,l}^n + {}^{(1)}\psi_{j-1,l}^n + {}^{(1)}\psi_{j,l+1}^n + {}^{(1)}\psi_{j,l-1}^n - 4\,{}^{(1)}\psi_{jl}^n), \end{aligned} \tag{7}$$

which supplies ${}^{(1)}\psi_{jl}^{n}$ provided that one knows ${}^{(1)}\psi_{jl}^{0}$ and ${}^{(1)}\psi_{jl}^{1}$. It is shown in Appendix 2 that one has a good approximation in

$$ {}^{(1)}\psi_{jl}^{0} = iK_0\psi_{jl}^{0}\,; \qquad {}^{(1)}\psi_{jl}^{1} = iK_0\psi_{jl}^{1}\,. \tag{7'} $$

2.3. *Numerical Tests of Eq.* (3)

We have used the difference equation (3) for investigating the propagation of a Gaussian laser beam of wavelength 10.6 μm with the data

$$ K_0 = 5.927533310 \times 10^5\ \mathrm{m}^{-1}; \qquad \psi^0(x, y) = \frac{1}{\pi^{1/2}} \exp\left(-\frac{x^2+y^2}{a^2}\right); \qquad a = 0.1\ \mathrm{m}. $$

The computations were made as far as 5 km with a step size $h = 50$ m and with a transverse step size $h_\rho = 3$ cm, so that one has

$$ \cos K_0 h = 0.67498; \qquad \sin K_0 h = 0.73783; $$
$$ b = 0.093724; \qquad \cos K_0 h + 4b \sin K_0 h = 0.95152. $$

Tables I, II, and III in Appendix 3 contain the results for 0.1, 2, and 5 km, respectively. Since the problem has a cylindrical symmetry around the Oz axis, it is sufficient to give the values of Re $\psi(r)$ and Im $\psi(r)$ on a square grid in one quadrant of the transverse plane. The dimension of the useful square is 15 $\times$ 15 cm. The values of $\bar{\psi}(r)\,\psi(r)$ are also given.

Eqs. (7) and (7′) were used for calculating the integral (6), and this integral is actually constant with good precision; the exact value is 1.185×10^4. Moreover, as a consequence of (6), the transverse energy $\int_{z=\mathrm{const}} \bar{\psi}(r)\,\psi(r)\,dx\,dy$ is conserved, which also shows in Tables I, II, and III (exact value, 10^{-2}).

It was interesting to check the stability condition (5) and we made the same calculations with $h_\rho = 2$ cm instead of $h_\rho = 3$ cm, which leads to $\cos K_0 h + 4b \sin K_0 h = 1.23$ so that condition (5) is not satisfied. We obtained excellent results as far as 2.5 km, and then there appeared an instability growing in two or three steps. Thus, one must be sure to satisfy (5).

In most of the work on laser beam propagation, people use a parabolic approximation (valid under paraxial conditions) of the Helmholtz equation which is the Schrodinger-like equation previously mentioned:

$$ 2iK_0 \frac{\partial \varphi(r)}{\partial z} + \Delta_\perp \varphi(r) = 0. \tag{8} $$

For the boundary data

$$ \varphi^0(x, y) = \frac{1}{\pi^{1/2}} \exp\left(-\frac{x^2+y^2}{2}\right), $$

Eq. (8) has the analytical solution

$$\varphi(\xi, \eta, \zeta) = \frac{1}{1 + i\zeta} \frac{1}{\pi^{1/2}} \exp\left\{- \frac{\zeta^2 + \eta^2}{2(1 + \zeta^2)} - i\, \frac{\xi^2 + \eta^2}{2} \times \frac{\zeta}{1 + \zeta^2}\right\},$$

with

$$\xi = x/a, \qquad \eta = y/a, \qquad \zeta = z/K_0 a^2,$$

and the paraxial approximation is $\hat{\psi}(r) = e^{iK_0 z}\varphi(r)$.

The computations of $\hat{\psi}(r)$ were made for the same conditions as those used previously, and they are given in Tables IV, V, and VI. The comparison between Tables I–III and IV–VI shows good agreement, especially for the energy density $\bar{\psi}\psi$.

3. Paraxial Approximation to the Solution of Eq. (1)

When one looks for approximate solutions of Eq. (1), one encounters new difficulties, although they are similar to those found with the Taylor series. A perturbative series can be formally written as

$$\psi_\epsilon(r) = \sum_{j=0}^{\infty} (-1)^j (\epsilon K_0^2)^j [(\Delta + K_0^2)^{-1} \mu(r)]^j \psi_0(r); \qquad (\Delta + K_0^2)\, \psi_0(r) = 0.$$

We were unable to find a satisfactory method for solving Eq. (1) without making further assumptions concerning the perturbation $\mu(r)$, and from now on, we assume the following inequalities to be valid:

$$| K_0^2 \mu^l(r)| \gg | \Delta_\perp \mu^l(r)|, \qquad | K_0^2 \mu^l(r)| \gg h_\rho^{-1} | \nabla_\perp \mu^l(r)|, \qquad l = 1, 2,..., \tag{9}$$

with

$$| \nabla_\perp \mu^l(r)| = \sup \left\{\left| \frac{\partial \mu^l(r)}{\partial x} \right|, \left| \frac{\partial \mu^l(r)}{\partial y} \right|\right\}, \qquad l = 1, 2,...,$$

where h_ρ is the transverse step, as previously.

In Appendix 4, we prove that if conditions (9) are fulfilled, the approximate solutions of Eq. (1) are the same as the approximate solutions of Eq. (2) with $K = K_0(1 + \epsilon\mu(r))$ instead of K_0. For the paraxial approximation of Eq. (1), conditions (9) do not intervene, and one has instead of (3):

$$\psi_{jl}^{n+1} + \psi_{jl}^{n-1} = 2 \cos K_{jl}^n h \psi_{jl}^n - b_{jl}^n \sin K_{jl}^n h \{\psi_{j+1,l}^n + \psi_{j-1,l}^n + \psi_{j,l+1}^n + \psi_{j,l-1}^n - 4\psi_{jl}^n\}, \tag{10}$$

with

$$K_{jl}^n = K_0\{1 + \epsilon\mu(jh_\rho, lh_\rho, nh)\}, \qquad b_{jl}^n = h/K_{jl}^n h_\rho^2. \tag{10'}$$

We tested (10) for the same conditions as those used previously, with

$$\mu(r) = \frac{1}{2}\left(1 + \operatorname{erf}\frac{x}{a_x}\right)\exp\left(-\frac{y^2}{a_y^2}\right); \qquad a_x = 100 \text{ cm}, \quad a_y = 10 \text{ cm}, \quad \epsilon = 10^{-7}, \tag{11}$$

which corresponds to a simulation of thermal blooming [7]. We found good results (that is, integral (6) is constant) as far as 5 km. In fact the value of integral (6)oscillates around the exact value. When ϵ is changed into $-\epsilon$, the results are good only to 2 km.

4. Conclusion

The results of Appendix 3 prove that algorithm (3) is especially interesting, since it is an explicit scheme which, as a consequence, is simple and fast but also has a good stability provided that the rather unrestrictive condition (5) is satisfied. Also, as suggested by the results for case (11) of laser beam propagation in a medium with variable index, algorithm (10) should generally be better (provided $K(r)$ varies slowly) than the usual difference equation used to solve an equation similar to (8):

$$2iK_0 \frac{\partial \varphi(r)}{\partial z} + \Delta_\perp \varphi(r) + \epsilon K_0^2 \mu(r)\, \varphi(r) = 0,$$

which leads to many well-known difficulties.

Appendix 1

We assume that $\psi(r)$ is as smooth as necessary to ensure the validity of all our expansions. Then the Taylor series expansion of $\psi^{n+1} + \psi^{n-1}$ is

$$\psi^{n+1} + \psi^{n-1} = 2 \sum_{p=0}^{\infty} (-1)^p \frac{h^{2p}}{(2p)!} \sum_{j=0}^{p} \binom{p}{j} K_0^{2(p-j)} \Delta_\perp{}^j \psi^n,$$

with $\binom{p}{j} = p!/j!\,(p-j)!$, where $\Delta_\perp{}^j$ is the jth iterated operator $\Delta_\perp{}^j = \Delta_\perp \Delta_\perp^{j-1}$. Rearranging terms with respect to increasing powers of $\Delta_\perp{}^j \psi^n$ gives

$$\psi^{n+1} + \psi^{n-1} = 2 \sum_{j=0}^{\infty} a_j \Delta_\perp{}^j \psi^n, \tag{A1}$$

$$a_j = h^{2j} \sum_{p=j}^{\infty} \frac{(-1)^p}{(2p)!} \binom{p}{j} (K_0 h)^{2(p-j)}. \tag{A2}$$

Lemma 1. *For every nonnegative integer j, one has*

$$a_j = -\frac{h^{2j}}{2(j!)} \frac{d^{j-1}}{dx^{2(j-1)}} \frac{\sin x}{x}, \qquad x = K_0 h. \tag{A3}$$

To prove (A3), we start from definition (A2)

$$a_j = \frac{h^{2j}}{j!} \sum_{p=j}^{\infty} \frac{(-1)^p\, p!}{(p-j)!\,(2p)!}\, x^{2(p-j)} = \frac{h^{2j}}{j!} (-1)^j \sum_{l=0}^{\infty} (-1)^l \frac{(l+j)!}{l!\,(2l+2j)!}\, x^{2l}. \quad \text{(A4)}$$

Let $f(x)$ be the sum on the right-hand side of (A4)

$$f(x) = \sum_{l=0}^{\infty} (-1)^l \frac{(l+j)!}{l!\,(2l+2j)!}\, x^{2l} = \sum_{l=0}^{\infty} (-1)^l \frac{(l+1)(l+2)\cdots(l+j)}{(2l+2j)!}\, x^{2l},$$

and using the new variable $u = x^2$:

$$\begin{aligned} f(u) &= \sum_{l=0}^{\infty} (-1)^l \frac{(l+1)(l+2)\cdots(l+j)}{(2l+2j)!}\, u^l \\ &= \frac{1}{2} \sum_{l=0}^{\infty} (-1)^l \frac{(l+1)(l+2)\cdots(l+j-1)}{(2l+2j-1)!}\, u^l \\ &= \frac{1}{2} \frac{d^{j-1}}{du^{j-1}} \left(\sum_{l=0}^{\infty} (-1)^l \frac{u^{l+j-1}}{(2l+2j-1)!} \right); \end{aligned}$$

that is,

$$\begin{aligned} (-1)^j f(x) &= \frac{1}{2} \frac{d^{j-1}}{dx^{2(j-1)}} \left\{ \frac{(-1)^j}{x} \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l+2j-1}}{(2l+2j-1)!} \right\} \\ &= \frac{1}{2} \frac{d^{j-1}}{dx^{2(j-1)}} \left\{ \frac{1}{x} \left[\sum_{k=0}^{j-2} (-1)^k \frac{x^{2k+1}}{(2k+1)!} - \sin x \right] \right\} \\ &= -\frac{1}{2} \frac{d^{j-1}}{dx^{2(j-1)}} \frac{\sin x}{x}. \end{aligned}$$

Inserting this result into (A4) gives (A3); this completes the proof.

COROLLARY. *Using the definition of the spherical Bessel functions of the first kind,*

$$j_l(x) = x^l \left(-\frac{1}{x} \frac{\partial}{\partial x} \right)^l \frac{\sin x}{x}; \qquad l = 0, 1, 2, \ldots,$$

one easily proves the relation

$$a_l = \frac{(-1)^l}{l!\, 2^l} \frac{h^{l+1}}{K_0^{l+1}}\, j_{l-1}(K_0 h); \qquad l = 1, 2, \ldots. \quad \text{(A5)}$$

So Eq. (A1) *becomes*

$$\psi^{n+1} + \psi^{n-1} = 2 \left(a_0 + \sum_{l=1}^{\infty} \frac{(-1)^l\, h^{l+1}}{l!\, 2^l K_0^{l-1}}\, j_{l-1}(K_0 h)\, \Delta_\perp{}^l \right) \psi^n. \quad \text{(A6)}$$

To discuss the conditions for which this series is convergent, we need the following lemma:

LEMMA 2. *The coefficients a_j satisfy the relations*

$$\frac{d}{dh}\left(\frac{2ja_j}{h}\right) = -a_{j-1}; \tag{A7a}$$

$$\frac{2ja_j}{h} = -\int a_{j-1}\,dh. \tag{A7b}$$

Let us prove (A7b). *The proof of* (A7a) *is similar. From* (A4), *one has*

$$a_{j-1} = \frac{1}{(j-1)!}\sum_{p=j-1}^{\infty} \frac{(-1)^p\, p!\, K_0^{2(p-j+1)} h^{2p}}{(p-j+1)!\,(2p)!};$$

integrating this expression with respect to h gives

$$\int a_{j-1}\,dh = \frac{1}{(j-1)!}\sum_{p=j-1}^{\infty} (-1)^p \frac{p!\, K_0^{2(p-j+1)} h^{2p+1}}{(p-j+1)!\,(2p)!\,(2p+1)},$$

or with the new index $q = p+1$:

$$\int a_{j-1}\,dh = \frac{-1}{(j-1)!}\sum_{q=j}^{\infty} (-1)^q \frac{(q-1)!\, K_0^{2(q-j)} h^{2q}}{(q-j)!\,(2q-2)!\,(2q-1)}$$

$$= \frac{-2}{h(j-1)!}\sum_{q=j}^{\infty} (-1)^q \frac{q!\, K_0^{2(q-j)} h^{2q}}{(q-j)!\,(2q)!} = \frac{-2j}{h}\, a_j\,.$$

This lemma has an important corollary.

COROLLARY. *One has $a_j = O(h^j/K_0^{\,j})$. In fact, using $x = K_0 h$, one can write a_j as*

$$a_j = \frac{1}{j!\,K_0^{2j}}\sum_{p=j}^{\infty} (-1)^p \frac{p!\, x^{2(p+j)}}{(p-j)!\,(2p)!},$$

so that Eq. (A7b) *becomes $(1/h)\,2K_0\, ja_j = -\int a_{j-1}\,dx$; thus, if $a_{j-1} = O(h^{j-1}/K_0^{j-1})$, this last relation implies $a_j = O(h^j/K_0^{\,j})$, and since $a_1 = O(h/K_0)$, this completes the proof.*

Now, to discuss the convergence of (A6), it is useful to introduce the dimensionless quantities

$$x = h_\rho \xi, \qquad y = h_\rho \eta, \qquad b = h/K_0 h_\rho^{\,2},$$

where h_ρ has dimension length. In terms of ξ, η, the transverse Laplacian becomes

$$\Delta_\perp \psi^n = \frac{1}{h_\rho^2}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\psi^n = \frac{1}{h_\rho^2}\tilde{\Delta}_\perp \psi^n.$$

Then, using this expression and the previous corollary, we see that $a_j \Delta_\perp^j = O(b^j)$, which leads to the following theorem:

THEOREM. *If $\tilde{\Delta}_\perp^j \psi^n$ is bounded for $j = 1, 2, \ldots$, then the series* (A1) *converges for* $b < 1$.

Equation (3) corresponds to the first two terms of (A6), and the next approximation is

$$\psi^{n+1} + \psi^{n-1} = 2\cos K_0 h \psi^n - b \sin K_0 h \tilde{\Delta}_\perp \psi^n$$
$$- \frac{b^2}{4}\left(\cos K_0 h - \frac{\sin K_0 h}{K_0 h}\right)\tilde{\Delta}_\perp^2 \psi^n + O(b^3).$$

APPENDIX 2

Here we discuss the initial conditions (7′) for Eq. (7). Let us first consider ${}^{(1)}\psi_{jl}^1$. Since $\psi_{jl}^1 = e^{iK_0 h}\varphi_{jl}^1$, one has

$${}^{(1)}\psi_{jl}^1 = iK_0 e^{iK_0 h}\varphi_{jl}^1 + e^{iK_0 h}\, {}^{(1)}\varphi_{jl}^1,$$

but ${}^{(1)}\varphi_{ji}^1 = (i/2K_0)\,\Delta_\perp \varphi_{ji}^1$, since $\varphi(r)$ is a solution of the Schrodinger equation $2iK_0(\partial\varphi(r)/\partial z) + \Delta_\perp \varphi(r) = 0$; thus:

$${}^{(1)}\psi_{jl}^{(1)} = iK_0 \psi_{jl}^1\left(1 + \frac{1}{2K_0^2}\frac{\Delta_\perp \varphi_{jl}^1}{\varphi_{jl}^1}\right) = iK_0 \psi_{jl}^1\left(1 + \frac{1}{2K_0^2 h_\rho^2}\frac{\tilde{\Delta}_\perp \varphi_{jl}^1}{\varphi_{jl}^1}\right),$$

and since $K_0^2 h_\rho^2 \gg |\tilde{\Delta}_\perp \varphi_{ji}^1| / |\varphi_{ji}^1|$, one finally has

$${}^{(1)}\psi_{jl}^1 = iK_0 \psi_{jl}^1.$$

To obtain ${}^{(1)}\varphi_{ji}^0$, we start with the relation given in the Introduction,

$$\psi(r) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_{z=0} \psi^0(\rho')\frac{e^{iK_0 R}}{R}\,dx'\,dy',$$

which leads to

$$\frac{\partial \psi(r)}{\partial z} = -\frac{1}{2\pi} \frac{\partial^2}{\partial z^2} \int_{z=0} \psi^0(\rho') \frac{e^{iK_0 R}}{R} \, dx' \, dy',$$

and, after some easy computations, one has

$$^{(1)}\psi^0(\rho) = iK_0 \psi^0(\rho) + \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty \frac{\partial \psi^0}{\partial t} (x + t\cos\theta, y + t\sin\theta) \frac{e^{iK_0 t}}{t} \, dt. \quad \text{(A8)}$$

For $K_0 \sim 10^5$, one may neglect the second term on the right-hand side of (A8), which reduces to

$$^{(1)}\psi^0_{jl} = iK_0 \psi^0_{jl} \,. \quad \text{(A9)}$$

To check whether (A9) is a good approximation of (A8), one considers the case of a Gaussian beam with $\psi^0(\rho) = \exp(-\rho^2)$; then Eq. (A8) becomes

$$^{(1)}\psi^0(\rho) = e^{-\rho^2} \left\{ iK_0 - 2 \int_0^\infty e^{-(t^2 - iK_0 t)} \left[I_0(2t\rho) - \frac{\rho}{t} I_1(2t\rho) \right] dt \right\}$$

with

$$I_0(z) = \frac{1}{\pi} \int_0^\pi \operatorname{ch}(z\cos\theta) \, d\theta; \qquad I_1(z) = \frac{1}{\pi} \int_0^\pi \operatorname{sh}(z\cos\theta) \cos\theta \, d\theta,$$

and one has

$$\left| \int_0^\infty e^{-(t^2 - iK_0 t)} \left(I_0(2t\rho) - \frac{\rho}{t} I_1(2t\rho) \right) dt \right| \leqslant \int_0^\infty e^{-t^2} \left| I_0(2t\rho) - \frac{\rho}{t} I_1(2t\rho) \right| dt.$$

We performed the integration on the right-hand side of the last inequality under the conditions of the numerical test of Section 2.3, and we found that this integral, which depends on ρ, has a value between 1 and 300 much less than K_0. This justifies the use of (A9).

Appendix 3

TABLE I

Solution of the Helmholtz Equation[a]
(D = 100 m)

y (cm)	x (cm): 0	3	6	9	12	15
			Re ψ			
0	−0.0407	−0.0389	−0.0340	−0.0271	−0.0198	−0.0132
3	−0.0389	−0.0372	−0.0325	−0.0259	−0.0189	−0.0126
6	−0.0340	−0.0325	−0.0284	−0.0226	−0.0165	−0.0110
9	−0.0271	−0.0259	−0.0226	−0.0181	−0.0132	−0.00878
12	−0.0198	−0.0189	−0.0165	−0.0132	−0.00960	−0.00641
15	−0.0132	−0.0126	−0.0110	−0.00878	−0.00641	−0.00428
			Im ψ			
0	0.563	0.537	0.469	0.373	0.271	0.179
3	0.537	0.513	0.448	0.356	0.258	0.171
6	0.469	0.448	0.390	0.310	0.225	0.149
9	0.373	0.356	0.310	0.247	0.179	0.119
12	0.271	0.258	0.225	0.179	0.130	0.0862
15	0.179	0.171	0.149	0.119	0.0862	0.0571
			$\bar{\psi} \cdot \psi$			
0	0.318625	0.289882	0.221117	0.139863	0.073833	0.032215
3	0.289882	0.264553	0.201760	0.127407	0.066921	0.029400
6	0.221117	0.201760	0.152907	0.096611	0.050897	0.022322
9	0.139863	0.127407	0.096611	0.061337	0.032215	0.014238
12	0.073833	0.066921	0.050897	0.032215	0.016992	0.007472
15	0.032215	0.029400	0.022322	0.014238	0.007472	0.003279

[a] Transverse energy flow : $\left| \int \left(\bar{\psi} \cdot \frac{\partial \psi}{\partial z} - \psi \cdot \frac{\partial \bar{\psi}}{\partial z} \right) dxdy \right| = 1.165 \times 10^4$.

Transverse energy: $\left| \int \bar{\psi} \cdot \psi \, dxdy \right| = 0.00983$.

TABLE II

Solution of the Helmholtz Equation[a]
(D = 2000 m)

y (cm)	x (cm) 0	3	6	9	12	15
			Re ψ			
0	0.0623	0.0532	0.0295	0.342×10^{-4}	−0.0252	−0.0397
3	0.0532	0.0447	0.0226	−0.00455	−0.0276	−0.0404
6	0.0295	0.0226	0.00507	−0.0162	−0.0334	−0.0419
9	0.342×10^{-4}	−0.00455	−0.0162	−0.0297	−0.0394	−0.0423
12	−0.0252	−0.0276	−0.0334	−0.0394	−0.0422	−0.0401
15	−0.0397	−0.0404	−0.0419	−0.0423	−0.0401	−0.0349
			Im ψ			
0	0.531	0.511	0.453	0.369	0.276	0.187
3	0.511	0.491	0.435	0.354	0.264	0.179
6	0.453	0.435	0.385	0.313	0.232	0.156
9	0.369	0.354	0.313	0.253	0.186	0.124
12	0.276	0.264	0.232	0.186	0.136	0.0893
15	0.187	0.179	0.156	0.124	0.0893	0.0575
			$\bar{\psi} \cdot \psi$			
0	0.2863	0.2637	0.20603	0.1365	0.07663	0.03644
3	0.2637	0.24287	0.18973	0.12566	0.07053	0.033528
6	0.20603	0.18973	0.14814	0.09804	0.05496	0.026099
9	0.1365	0.12566	0.09804	0.0648	0.03627	0.01719
12	0.07664	0.07053	0.05496	0.03627	0.02026	0.00959
15	0.03644	0.033528	0.026099	0.01719	0.00959	0.0045356

[a] Transverse energy flow: $\left| \int \left(\bar{\psi} \cdot \frac{\partial \psi}{\partial z} - \psi \cdot \frac{\partial \bar{\psi}}{\partial z} \right) dxdy \right| = 1.163 \times 10^{4}$.

Transverse energy: $\left| \int \bar{\psi} \cdot \psi \, dxdy \right| = 0.00999$.

TABLE III

Solution of the Helmholtz Equation[a]
($D = 5000$ m)

y (cm)	x (cm) 0	3	6	9	12	15
			Re ψ			
0	0.362	0.347	0.304	0.239	0.164	0.0903
3	0.347	0.332	0.290	0.228	0.155	0.0834
6	0.304	0.290	0.251	0.194	0.128	0.0643
9	0.239	0.228	0.194	0.145	0.0897	0.0374
12	0.164	0.155	0.128	0.0897	0.0472	0.00872
15	0.0903	0.0834	0.0643	0.0374	0.00872	−0.0156
			Im ψ			
0	0.243	0.244	0.246	0.244	0.231	0.203
3	0.244	0.245	0.246	0.243	0.228	0.200
6	0.246	0.246	0.245	0.237	0.219	0.189
9	0.244	0.243	0.237	0.225	0.203	0.170
12	0.231	0.228	0.219	0.203	0.178	0.144
15	0.203	0.200	0.189	0.170	0.144	0.112
			$\bar{\psi} \cdot \psi$			
0	0.190093	0.179945	0.152932	0.116657	0.080257	0.049363
3	0.179945	0.170249	0.144616	0.111033	0.076009	0.046956
6	0.152932	0.144616	0.123026	0.093805	0.064345	0.039855
9	0.116657	0.111033	0,093805	0.071650	0.049255	0.030299
12	0.080257	0.076009	0.064345	0.049255	0.033912	0.020812
15	0.049363	0.046956	0.039855	0.030299	0.020812	0.012787

[a] Transverse energy flow: $\left| \int \left(\bar{\psi} \cdot \frac{\partial \psi}{\partial z} - \psi \cdot \frac{\partial \bar{\psi}}{\partial z} \right) dxdy \right| = 1.179 \cdot 10^4.$

Transverse energy: $\left| \int \bar{\psi} \cdot \psi \, dxdy \right| = 0.0100.$

TABLE IV

Solution of the Schrödinger Equation
(D = 100 m)

y (cm)	x (cm) 0	3	6	9	12	15
	Re ψ					
0	−0.0406	−0.0384	−0.0325	−0.0245	−0.0164	−0.00972
3	−0.0384	−0.0363	−0.0307	−0.0232	−0.0155	−0.00916
6	−0.0325	−0.0307	−0.0259	−0.0195	−0.0130	−0.00765
9	−0.0245	−0.0232	−0.0195	−0.0146	−0.00972	−0.00565
12	−0.0164	−0.0155	−0.0130	−0.00972	−0.00638	−0.00365
15	−0.00972	−0.00916	−0.00765	−0.00565	−0.00365	−0.00203
	Im ψ					
0	0.563	0.538	0.470	0.375	0.274	0.183
3	0.538	0.514	0.449	0.359	0.262	0.175
6	0.470	0.449	0.393	0.314	0.229	0.153
9	0.375	0.359	0.314	0.251	0.183	0.122
12	0.274	0.262	0.229	0.183	0.133	0.0891
15	0.183	0.175	0.153	0.122	0.0891	0.0595
	$\bar{\psi} \cdot \psi$					
0	0.318617	0.290919	0.221956	0.141225	0.075345	0.033584
3	0.290919	0.265514	0.202543	0.129419	0.068884	0.030709
6	0.221956	0.202543	0.155120	0.098976	0.052610	0.023468
9	0.141225	0.129419	0.098976	0.063214	0.033583	0.014916
12	0.0075345	0.068884	0.052610	0.033583	0.017730	0.007952
15	0.033584	0.030709	0.023468	0.014916	0.007952	0.003544

TABLE V

Solution of the Schrödinger Equation
($D = 2000$ m)

	x (cm)					
y (cm)	0	3	6	9	12	15
			Re ψ			
0	0.0629	0.0674	0.0780	0.0885	0.0923	0.0862
3	0.0674	0.0714	0.0808	0.0898	0.0922	0.0851
6	0.0780	0.0808	0.0871	0.0920	0.0907	0.0813
9	0.0885	0.0898	0.0920	0.0918	0.0862	0.0743
12	0.0923	0.0922	0.0907	0.0862	0.0772	0.0639
15	0,0862	0.0851	0.0813	0.0743	0.0639	0.0508
			Im ψ			
0	0.531	0.509	0.448	0.361	0.264	0.175
3	0.509	0.488	0.429	0.345	0.253	0.167
6	0.448	0.429	0.377	0.302	0.220	0.144
9	0.361	0.345	0.302	0.241	0.175	0.113
12	0.264	0.253	0.220	0.175	0.125	0.0795
15	0.175	0.167	0.144	0.113	0.0795	0.0494
			$\bar{\psi} \cdot \psi$			
0	0.28577	0.26359	0.20685	0.1381	0.07844	0.03791
3	0.26359	0.24313	0.190795	0.12738	0.072355	0.034966
6	0.20685	0.190795	0.14972	0.09996	0.05678	0.02744
9	0.1381	0.12738	0.09996	0.0667	0.03644	0.01832
12	0.07844	0.072355	0.05678	0.03644	0.02153	0.01040
15	0.03791	0.034966	0.02744	0.01832	0.01040	0.00503

TABLE VI

Solution of the Schrödinger Equation
($D = 5000$ m)

y (cm)	x (cm) 0	3	6	9	12	15
	Re ψ					
0	0.355	0.351	0.338	0.313	0.274	0.223
3	0.351	0.347	0.333	0.308	0.269	0.218
6	0.338	0.331	0.318	0.291	0.252	0.201
9	0.313	0.308	0.291	0.263	0.223	0.174
12	0.274	0.269	0.252	0.223	0.185	0.140
15	0.223	0.218	0.201	0.174	0.140	0.101
	Im ψ					
0	0.245	0.231	0.191	0.134	0.0696	0.0109
3	0.231	0.217	0.179	0.123	0.0619	0.00581
6	0.191	0.179	0.144	0.0949	0.0405	−0.00801
9	0.134	0.123	0.0949	0.0545	0.0109	−0.0264
12	0.0696	0.0619	0.0405	0.0109	−0.0197	−0.0442
15	0.0109	0.00581	−0.00801	−0.0264	−0.0442	−0.0561
	$\bar{\psi} \cdot \psi$					
0	0.18605	0.176562	0.150725	0.115925	0.079920	0.049848
3	0.176562	0.167498	0.141602	0.109993	0.076193	0.047558
6	0.150725	0.141602	0.121860	0.093687	0.065144	0.040465
9	0.115925	0.109993	0.093687	0.072139	0.049848	0.030973
12	0.079920	0.076193	0.065144	0.049848	0.034613	0.021554
15	0.049848	0.047558	0.040465	0.030973	0.021554	0.013348

Appendix 4

Here, we prove that expression (A6) with $K = K_0(1 + \epsilon\mu(r))$ instead of K_0 gives an approximate solution of (1) provided that inequalities (9) are satisfied.

Indeed, from (A1) and (A2), it is clear that one need only prove the relations

$$(\Delta_\perp + K^2) \cong \sum_{j=0}^{n} \binom{n}{j} K^{2j} \Delta_\perp^{n-j}, \qquad n = 2, 3, \ldots, \tag{A10}$$

which are not exact, since $\Delta_\perp$ and K^2 do not commute, but which are approximately valid when inequalities (9) are satisfied, as we shall prove.

Lemma 3. *If* (i) *the relation* (A10) *holds for* $n \leqslant n_0$, (ii) *inequalities* (9) *are satisfied, then relation* (A10) *is still valid for* $n_0 + 1$.

Starting with (A10), one has

$$(\Delta_\perp + K^2)^{n+1} = (\Delta_\perp + K^2)(\Delta_\perp + K^2)^n = \sum_{j=0}^{\infty} \binom{n}{j} (K^{2j+2}\Delta_\perp^{n-j} + \Delta_\perp(K^{2j}\Delta_\perp^{n-j})), \tag{A11}$$

but

$$K^{2j+2} = K_0^{2j+2} \sum_{l=0}^{j+1} \binom{j+1}{l} \epsilon^l \mu^l, \tag{A12a}$$

and

$$\begin{aligned}\Delta_\perp(K^{2j}\Delta_\perp^{n-j}) &= K^{2j}\Delta_\perp^{n+1-j} + 2\nabla_\perp K^{2j} \cdot \nabla_\perp \Delta_\perp^{n-j} + \Delta_\perp K^{2j} \Delta_\perp^{n-j} \\ &= K_0^{2j} \sum_{l=0}^{j} \binom{j}{l} \epsilon^l \{\mu^l \Delta_\perp^{n+1-j} + 2\nabla_\perp \mu^l \cdot \nabla_\perp \Delta_\perp^{n-j} + \Delta_\perp \mu^l \Delta_\perp^{n-j}\},\end{aligned} \tag{A12b}$$

with (A12a) and (A12b), Eq. (A11) becomes

$$\begin{aligned}(\Delta_\perp + K^2)^{n+1} = \sum_{j=0}^{n} \binom{n}{j} K_0^{2j} \sum_{l=0}^{j} \epsilon^l \Big\{ & K_0^2 \binom{j+1}{l} \mu^l \Delta_\perp^{n-j} \\ & + \binom{j}{l} \mu^l \Delta_\perp^{n+1-j} + 2\binom{j}{l} \nabla_\perp \mu^l \cdot \nabla_\perp \Delta_\perp^{n-j} \\ & + \binom{j}{l} \Delta_\perp \mu^l \Delta_\perp^{n-j} \Big\} + \sum_{j=0}^{n} \binom{n}{j} K_0^{2j+2} \epsilon^{j+1} \mu^{j+1} \Delta_\perp^{n-j},\end{aligned}$$

and, with a rearrangement of terms with respect to increasing powers of ϵ^l, this becomes

$$(\Delta + K^2)^{n+1} = \sum_{l=0}^{n} \epsilon^l \sum_{j=1}^{n} \binom{n}{j} K_0^{2j} \Big\{ K_0^2 \binom{j+1}{l} \mu^l \Delta_\perp^{n-j} + \binom{j}{l} \mu^l \Delta_\perp^{n+1-j} + 2 \binom{j}{l} \nabla_\perp \mu^l \cdot \nabla_\perp \Delta_\perp^{n-j} + \binom{j}{l} \Delta_\perp \mu^l \Delta_\perp^{n-j} \Big\} + \sum_{l=1}^{n} \epsilon^l \mu^l \binom{n}{l-1} K_0^{2l} \Delta_\perp^{n+1-l}.$$

Taking (9) into account, the coefficient of ϵ^l reduces to

$$\mu^l \sum_{j=l}^{n} \binom{n}{j} K_0^{2j} \Big\{ K_0^2 \binom{j+1}{l} \Delta_\perp^{n-j} + \binom{j}{l} \Delta_\perp^{n+1-j} \Big\} + \mu^l \binom{n}{l-1} K_0^{2l} \Delta_\perp^{n+1-l}, \qquad \text{(A13)}$$

but, if (A10) is valid for $n + 1$, one has

$$\begin{aligned}(\Delta_\perp + K^2)^{n+1} &= \sum_{j=0}^{n+1} \binom{n+1}{j} K_0^{2j} \sum_{l=0}^{j} \binom{j}{l} \epsilon^l \mu^l \Delta_\perp^{n+1-j} \\ &= \sum_{l=0}^{n+1} \epsilon^l \mu^l \sum_{j=l}^{n+1} \binom{j}{l} \binom{n+1}{j} K_0^{2j} \Delta_\perp^{n+1-j}\end{aligned}$$

so that the coefficient ϵ^l is

$$\mu^l \sum_{j=l}^{n+1} \binom{j}{l} \binom{n+1}{j} K_0^{2j} \Delta_\perp^{n+1-j} \qquad \text{(A14)}$$

Using the well-known relation $\binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p}$, one may easily show that expressions (A13) and (A14) are equal, which completes the proof.

COROLLARY. *Since Eq.* (A10) *is true for* $n = 1$, *it is also valid for any* n *provided that* (9) *is satisfied.*

As a consequence, expression (A6) with K instead of K_0 supplies approximations to the solution of Eq. (1).

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